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Exact Results on Equations of Motion in Vacuum String Field Theory

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Abstract

We prove some algebraic relations on the translationally invariant solutions and the lump solutions in vacuum string field theory. We show that up to the subtlety at the midpoint the definition of the half-string projectors of the known sliver solution can be generalized to other solutions. We also find that we can embed the translationally invariant solution into the matrix equation of motion with the zero mode.

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1 Introduction and summary

Since vacuum string field theory (VSFT) was proposed by Rastelli, Sen and Zwiebach [1] to describe the true vacuum realized after the decay of all the D-branes, various classical solutions have been constructed*. Some of them represent the translationally invariant background in the Neumann direction and others represent lump solutions in the Dirichlet direction. However, there has not been a full understanding of the whole moduli space of the solutions in VSFT†. In this paper, we shall present some exact results on the moduli space and the equations of motion in VSFT.

The problem of finding a translationally invariant solution of the squeezed state form (up to normalization)

$$|N\rangle = \exp\biggl(-\frac{1}{2}\sum_{m,n\geq 1} a_m^\dagger S_{mn} a_n^\dagger\biggr)|0\rangle, \tag{1.1}$$

to the matter part of the equation of motion $\Psi_m *_m \Psi_m = \Psi_m$ is equivalent to solving an ∞-dimensional matrix equation of motion (EOM) for the matrix S

$$S = V_0 + (V_+, V_-)(1 - S\mathcal{V})^{-1} S \begin{pmatrix} V_- \\ V_+ \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} V_0 & V_+ \\ V_- & V_0 \end{pmatrix}, \tag{1.2}$$

with the Neumann coefficient matrices $V_\alpha(\alpha = 0, \pm)$ constituting the three-string interaction vertex. One way to find a non-trivial solution [5] is to reexpress the EOM in terms of mutually commutative matrices $M_\alpha = CV_\alpha$ with $C_{mn} = (-1)^m \delta_{mn}$ being the twist matrix and solve algebraically the EOM, which is equivalent to (1.2),

$$T = M_0 + (M_+, M_-)(1 - T\mathcal{M})^{-1} T \begin{pmatrix} M_- \\ M_+ \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M_0 & M_+ \\ M_- & M_0 \end{pmatrix}, \tag{1.3}$$

for $T = CS$ under the assumption that T commutes with all the matrices M_α. The solution obtained in this way is called the sliver state. However, the general structure of the space of solutions to (1.3) without assuming the commutativity $[T, M_\alpha] = 0$ is unknown yet.

The structure of the moduli space is understood better in the boundary conformal field theory (BCFT) construction of the solutions. According to [6], all the solutions of the surface state ansatz are characterized by the following property: The boundary of the surface reaches the midpoint of the local coordinate and the wave functional is split into the left one and the right one.

Our first topic in this paper is to characterize the solutions by an algebraic property which corresponds to the above split property in the BCFT construction. What has been done in

*See [2] for reviews and references of VSFT.

†There have been some attempts to explore the moduli space of the solutions, for example [3, 4].

the algebraic construction so far is as follows. In [7], two matrices $\rho_\pm$ satisfying the projector conditions

$$\rho_+ + \rho_- = 1, \quad \rho_+^2 = \rho_+, \quad \rho_-^2 = \rho_-, \tag{1.4}$$

were defined out of the sliver solution T by

$$(\rho_+, \rho_-) = (M_+, M_-)(1 - T\mathcal{M})^{-1}, \tag{1.5}$$

and they are interpreted as the projectors onto the left/right halves of the strings. This interpretation is made possible by a beautiful property ((3.30) in [7] up to normalization), $|N, \boldsymbol{k}_1\rangle * |N, \boldsymbol{k}_2\rangle = |N, \rho_+\boldsymbol{k}_1 + \rho_-\boldsymbol{k}_2\rangle$, for $|N, \boldsymbol{k}\rangle = \exp(-\boldsymbol{a}^\dagger C\boldsymbol{k})|N\rangle$, which claims that, only the insertion of $\boldsymbol{k}$ in the right half of the first string and the left half of the second string matters in the final result.

Putting the BCFT characterization of solutions together with the above algebraic property of $\rho_\pm$, we are led to the following expectation. Though only the sliver state was considered in [7], from the argument of general solutions in BCFT we naturally expect that $\rho_\pm$ (1.5) satisfy the projector conditions (1.4) as long as T satisfies the EOM (1.3), even though T is not the sliver state, (or in other words, even though T does not commute with M_α). We shall prove it in sec. 2. Though we do not assume the commutativity $[T, M_\alpha] = 0$, we utilize $[M_\alpha, M_\beta] = 0$ freely. Note that the use of $[M_\alpha, M_\beta] = 0$ is only allowed up to the ambiguity of the midpoint [8]. As a simple application of our results, we shall construct the tachyon fluctuation around general classical solutions by solving the linearized equation of motion [9]. We find that the interpretation of the tachyon state as inserting momentum at the midpoint [10] is still valid for general solutions.

So far, we have characterized the translationally invariant solutions by an algebraic property. In addition to the translationally invariant solutions, there are also lump solutions in VSFT. Our next topic is the relation between these two kinds of solutions. In search of lump solutions [5], the zero mode oscillator $a_0 = \big(\sqrt{b}/2\big)p - \big(i/\sqrt{b}\big)x$, with b being a new parameter, was introduced by combining the center-of-mass coordinate x and momentum p to make an ansatz (up to normalization)

$$|\Xi_b\rangle = \exp\biggl(-\frac{1}{2}\sum_{m,n\geq 0} a_m^\dagger (S')_{mn} a_n^\dagger\biggr)|\Omega_b\rangle. \tag{1.6}$$

Here the new vacuum $|\Omega_b\rangle$ is defined to be annihilated by a_n including the zero mode and is related to the momentum eigenstate $|p\rangle$ by

$$|p\rangle = \exp\biggl(-\frac{1}{2}(a_0^\dagger)^2 + \sqrt{b}pa_0^\dagger - \frac{b}{4}p^2\biggr)|\Omega_b\rangle. \tag{1.7}$$

If we reexpress the string interaction vertex on the new vacuum $|\Omega_b\rangle$, the EOM (1.2) are replaced by an "$\infty+1$"-dimensional matrix equation of motion (EOM$'$) for S'

$$S' = V_0' + (V_+', V_-')(1 - S'\mathcal{V}')^{-1} S' \begin{pmatrix} V_-' \\ V_+' \end{pmatrix}, \quad \mathcal{V}' = \begin{pmatrix} V_0' & V_+' \\ V_-' & V_0' \end{pmatrix}, \tag{1.8}$$

where all the matrices V_α', S' are bigger than their cousins without primes by one row and column of the zero mode. The explicit forms of V_0', V_+' and V_-' are given by [5]

$$V_0' = \begin{pmatrix} (V_0')_{00} & (V_0')_{0n} \\ (V_0')_{m0} & (V_0')_{mn} \end{pmatrix} = \begin{pmatrix} 1 - 2b/(3\beta) & \sqrt{2b}\boldsymbol{v}_0^{\mathrm{T}}/\beta \\ \sqrt{2b}\boldsymbol{v}_0/\beta & V_0 - 2U_0/\beta \end{pmatrix}, \tag{1.9}$$

$$V_+' = \begin{pmatrix} (V_+')_{00} & (V_+')_{0n} \\ (V_+')_{m0} & (V_+')_{mn} \end{pmatrix} = \begin{pmatrix} b/(3\beta) & \sqrt{2b}\boldsymbol{v}_-^{\mathrm{T}}/\beta \\ \sqrt{2b}\boldsymbol{v}_+/\beta & V_+ - 2U_+/\beta \end{pmatrix}, \tag{1.10}$$

$$V_-' = \begin{pmatrix} (V_-')_{00} & (V_-')_{0n} \\ (V_-')_{m0} & (V_-')_{mn} \end{pmatrix} = \begin{pmatrix} b/(3\beta) & \sqrt{2b}\boldsymbol{v}_+^{\mathrm{T}}/\beta \\ \sqrt{2b}\boldsymbol{v}_-/\beta & V_- - 2U_-/\beta \end{pmatrix}, \tag{1.11}$$

with U_0, U_+ and U_- defined as

$$U_0 = \boldsymbol{v}_0\boldsymbol{v}_0^{\mathrm{T}} + \boldsymbol{v}_+\boldsymbol{v}_+^{\mathrm{T}} + \boldsymbol{v}_-\boldsymbol{v}_-^{\mathrm{T}}, \tag{1.12}$$

$$U_+ = \boldsymbol{v}_0\boldsymbol{v}_-^{\mathrm{T}} + \boldsymbol{v}_+\boldsymbol{v}_0^{\mathrm{T}} + \boldsymbol{v}_-\boldsymbol{v}_+^{\mathrm{T}}, \tag{1.13}$$

$$U_- = \boldsymbol{v}_0\boldsymbol{v}_+^{\mathrm{T}} + \boldsymbol{v}_+\boldsymbol{v}_-^{\mathrm{T}} + \boldsymbol{v}_-\boldsymbol{v}_0^{\mathrm{T}}, \tag{1.14}$$

and $\beta = 2V_{00} + b/2$. We can construct a non-trivial solution [5] similarly to the case of (1.2) by rewriting the EOM$'$ (1.8) in terms of mutually commutative matrices $M_\alpha' = C'V_\alpha'$ with $C_{mn}' = (-1)^m\delta_{mn}$ and finding a solution $T' = C'S'$ which commutes with all the matrices M_α'.

Note that the ansatz of (1.6) allows the non-trivial momentum dependence of the lump solutions. However, the translationally invariant solution with zero momentum (1.1) also fits in this framework as a special case. In fact, the translationally invariant solution (1.1) can be reexpressed in terms of $|\Omega_b\rangle$ as

$$|N\rangle = \exp\left(-\frac{1}{2}\boldsymbol{a}^\dagger S \boldsymbol{a}^\dagger - \frac{1}{2}(a_0^\dagger)^2\right)|\Omega_b\rangle, \tag{1.15}$$

with the help of (1.7) by setting $p = 0$, and it satisfies the squeezed state ansatz of (1.6). Hence, we naturally expect that we can embed the translationally invariant solution into the EOM$'$ (1.8). More explicitly, we shall prove in sec. 3 that S' defined by

$$S' = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}, \tag{1.16}$$

satisfies the EOM$'$ (1.8), if and only if S satisfies the EOM (1.2).

There is another construction of lump solutions using BCFT [11]. As a final topic in this paper, we shall investigate the equation of motion for them. The BCFT lump solution is

constructed by inserting on the surface two twist fields whose positions are parameterized by t. The oscillator representation of this construction was given explicitly by [12]

$$|D_t\rangle = \int dp \exp\biggl(-\frac{1}{2}\boldsymbol{a}^\dagger Q \boldsymbol{a}^\dagger - p\boldsymbol{l}^{\mathrm{T}}\boldsymbol{a}^\dagger\biggr)|p\rangle, \tag{1.17}$$

in terms of t-dependent quantities Q and $\boldsymbol{l}$. Using (1.7) we can reexpress (1.17) in the form of (1.6) and the corresponding S' is given by

$$S' = Q' - \frac{1}{2\alpha}\boldsymbol{l}'\boldsymbol{l}'^{\mathrm{T}}, \quad Q' = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}, \quad \boldsymbol{l}' = \begin{pmatrix} \sqrt{b} \\ \boldsymbol{l} \end{pmatrix}, \tag{1.18}$$

with $\alpha = b/4 + \log(2t)$. Since the BCFT lump solutions (1.17) satisfies the ansatz of (1.6), S' given by (1.18) has to satisfy the EOM′ (1.8). Though the EOM′ (1.8) apparently depends on b, it should be possible to reduce it into a b-independent form because originally both the solutions (1.17) and the three-string interaction vertex do not depend on b. This is our task in sec. 4.

We might carry out the following naive argument on the EOM′ (1.8) satisfied by S' of (1.18). Since we know that the b-dependence of the EOM′ (1.8) for S' of (1.18) is only apparent, let us take the limit $b \to \infty$ in (1.9)–(1.11) and (1.18),

$$V'_0 \to \begin{pmatrix} -1/3 & 0 \\ 0 & V_0 \end{pmatrix}, \quad V'_+ \to \begin{pmatrix} 2/3 & 0 \\ 0 & V_+ \end{pmatrix}, \quad V'_- \to \begin{pmatrix} 2/3 & 0 \\ 0 & V_- \end{pmatrix}, \quad S' \to \begin{pmatrix} -1 & 0 \\ 0 & Q \end{pmatrix}. \tag{1.19}$$

It turns out that the non-zero-mode components decouple from the zero-mode one and it seems that, by picking up the non-zero-mode components, Q satisfies the EOM (1.2) with S replaced by Q. However, this is incorrect. The reason is that the inverse of the zero-mode block of $1 - S'\mathcal{V}'$ is not well-defined. After detailed analysis given in sec. 4, we find that the equation of motion satisfied by the solution in the BCFT construction is (4.13), (4.14) and (4.15). The calculation is very similar to that in sec. 3 and we shall be brief in sec. 4. The same result can also be obtained by the b-independent calculation from the beginning (with b-independent expression (1.17) and b-independent star product) after integrating out the internal momentum.

To summarize, let us list up some lessons we have learned from our calculations in this paper. • We have characterized the solutions to the EOM (1.2) by an algebraic property that $\rho_\pm$ defined in (1.5) satisfy the projector conditions (1.4). • The EOM′ (1.8) has bigger moduli of solution than the EOM (1.2). In fact, if the EOM (1.2) is satisfied, we can always embed the solution into the solution of the EOM′ (1.8) by (1.16). • We have written down the b-independent form of the equation of motion satisfied by the solution in the BCFT construction. • Since the b-dependence of the star multiplication is superficial, the origin of the b-dependence which enters in the lump solution (1.6) in the algebraic construction [5] is mainly the assumption of the commutativity with M'_α.

We believe we have clarified the moduli space of the classical solutions to some extent. It is an important future work to understand the whole moduli space.

2 $\rho_\pm$ as projectors

In this section, we shall prove that $\rho_\pm$ defined in (1.5) satisfies the projector conditions (1.4) if only T satisfies the EOM (1.3). Let us begin with proving

$$\rho_+ + \rho_- = 1. \tag{2.1}$$

For this purpose, we first multiply the identity $(1-T\mathcal{M})^{-1}(1-T\mathcal{M}) = 1$ by $(1,1)^{\mathrm{T}}$ from the right and obtain

$$(1-T\mathcal{M})^{-1}\begin{pmatrix}1-T(1-M_-)\\1-T(1-M_+)\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}, \tag{2.2}$$

with the use of the relation

$$M_0 + M_+ + M_- = 1. \tag{2.3}$$

Then, multiplying (2.2) by (M_+, M_-) from the left, we obtain

$$(M_+,M_-)(1-T\mathcal{M})^{-1}(1-T)\begin{pmatrix}1\\1\end{pmatrix} + (M_+,M_-)(1-T\mathcal{M})^{-1}T\begin{pmatrix}M_-\\M_+\end{pmatrix} = M_+ + M_-. \tag{2.4}$$

Using the definition (1.5) of $\rho_\pm$ for the first term on the left hand side (LHS), the EOM (1.3) for the second one and the relation (2.3) for the right hand side (RHS), we have

$$(\rho_+ + \rho_-)(1-T) + (T - M_0) = 1 - M_0. \tag{2.5}$$

Therefore, (2.1) is proved.

Next we have to show either $\rho_+^2 = \rho_+$, $\rho_-^2 = \rho_-$, $\rho_+\rho_- = 0$ or

$$(\rho_+ - \rho_-)^2 = 1 \tag{2.6}$$

to prove that ρ_+ and ρ_- are actually the projectors. We find that the last one (2.6) is the easiest to prove. For this purpose, we rewrite the EOM (1.3) with $M_\pm = (1 - M_0 \pm M_1)/2$ into

$$T = M_0 + \frac{1}{2}(\rho_+ + \rho_-)T(1-M_0) - \frac{1}{2}(\rho_+ - \rho_-)TM_1. \tag{2.7}$$

Using the relation (2.1) we have just proved, we find a relation for $\rho_+ - \rho_-$,

$$(\rho_+ - \rho_-)TM_1 = 2M_0 - T(1+M_0). \tag{2.8}$$

Multiplying (2.8) by $\rho_+ - \rho_-$ from the left and by M_1 from the right, we find

$$(\rho_+ - \rho_-)^2 TM_1^2 = 2(\rho_+ - \rho_-)M_0M_1 - \big(2M_0 - T(1+M_0)\big)(1+M_0), \tag{2.9}$$

where we have used the commutativity $[M_0, M_1] = 0$ and (2.8) again. Similarly, multiplying (1.5) by $(1, -1)^{\mathrm{T}}$ from the right and using $(1 - T\mathcal{M})^{-1} = 1 + (1 - T\mathcal{M})^{-1} T\mathcal{M}$ and

$$\mathcal{M} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{2}(1 - 3M_0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \frac{1}{2} M_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2.10}$$

we obtain

$$\rho_+ - \rho_- = M_1 - \frac{1}{2}(\rho_+ - \rho_-) T (1 - 3M_0) - \frac{1}{2} T M_1. \tag{2.11}$$

Therefore we have

$$(\rho_+ - \rho_-)\big(2 + T(1 - 3M_0)\big) = 2M_1 - TM_1. \tag{2.12}$$

Multiplying (2.12) by $\rho_+ - \rho_-$ from the left again, we find, with the use of (2.8),

$$(\rho_+ - \rho_-)^2 \big(2 + T(1 - 3M_0)\big) = 2(\rho_+ - \rho_-) M_1 - \big(2M_0 - T(1 + M_0)\big). \tag{2.13}$$

Then, the combination $\big[(2.9) - (2.13) \times M_0\big]$ gives a relation without the terms linear in $\rho_+ - \rho_-$:

$$(\rho_+ - \rho_-)^2 \Big[T M_1^2 - \big(2 + T(1 - 3M_0)\big) M_0 \Big] = -2M_0 + T(1 + M_0). \tag{2.14}$$

The quantity on the RHS of (2.14) and that in the square parentheses on the LHS are found to be identical to each other[‡] with the use of $M_1^2 = (1 - M_0)(1 + 3M_0)$. This shows (2.6) and completes our proof. Note that our calculation applies similarly to the case with the zero mode by replacing all the quantities by those with primes.

Having shown that $\rho_\pm$ are projectors without using the commutativity $[T, M_\alpha] = 0$, as a simple application let us construct the tachyon state for any translationally invariant solution of the form (1.1)[§], by solving the linearized equation of motion for a general T. The linearized equation of motion for the tachyon mode $\exp\big(-\sqrt{2}\boldsymbol{t}^{\mathrm{T}} \boldsymbol{a}^\dagger \cdot p + ip \cdot \hat{x}\big) |N\rangle$ carrying the center-of-mass momentum p leads to the vector equation for $\boldsymbol{t}$ [9]:

$$(1 - \rho_-) \boldsymbol{t} = \boldsymbol{v}_0 - \boldsymbol{v}_+ + (\rho_+, \rho_-) T \begin{pmatrix} \boldsymbol{v}_+ - \boldsymbol{v}_- \\ \boldsymbol{v}_- - \boldsymbol{v}_0 \end{pmatrix}. \tag{2.15}$$

Summing up (2.15) and its twist conjugate with the use of $\boldsymbol{v}_\pm = (-\boldsymbol{v}_0 \pm \boldsymbol{v}_1)/2$, we find

$$(2 - \rho_+ - \rho_-) \boldsymbol{t} = 3\boldsymbol{v}_0 - \frac{3}{2}(\rho_+ + \rho_-) T \boldsymbol{v}_0 + \frac{3}{2}(\rho_+ - \rho_-) T \boldsymbol{v}_1. \tag{2.16}$$

[‡]These quantities vanish at the midpoint $M_0 = -1/3$. Therefore, besides the commutativity $[M_0, M_1] = 0$, our proof also suffers the midpoint ambiguity [8] in this sense. In fact, the existence of the eigenvalue 1/2 of $\rho_\pm$ is necessary for reproducing the massive open string states around the translationally invariant solution [13].

[§]See also [4], where they study the linearized equation of motion for a class of solutions called dressed slivers.

By further using (2.1), (2.8) and the following expression of $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ [8, 14]

$$\boldsymbol{v}_0 = -\frac{1}{3}(1+3M_0)\frac{\boldsymbol{\xi}(\pi/2)}{\sqrt{2}}, \quad \boldsymbol{v}_1 = M_1\frac{\boldsymbol{\xi}(\pi/2)}{\sqrt{2}}, \tag{2.17}$$

with $\xi_n(\sigma) = \sqrt{2/n}\cos n\sigma$, we easily solve (2.16) for $\boldsymbol{t}$,

$$\boldsymbol{t} = -(1+T)\frac{\boldsymbol{\xi}(\pi/2)}{\sqrt{2}}, \tag{2.18}$$

without using the commutativity $[T, M_\alpha] = 0$. However, we still have to check that when multiplied by ρ_- the RHS of (2.15) vanishes. Equivalently, we can show that the difference between (2.15) and its twist conjugate gives the same result as (2.18) for $\boldsymbol{t}$ with the help of (2.17) and (2.12). This completes our solution to the linearized equation of motion. The expression (2.18) can be interpreted as inserting momentum at the midpoint on the classical solution. Hence, the interpretation of the midpoint momentum insertion [10] is still valid even for general solutions of the EOM (1.2).

3 EOM′ for the translationally invariant solution

In this section, we shall report on the relation between the EOM (1.2) and the EOM′ (1.8). Namely, we shall prove the equivalence between the condition that S' given by (1.16) satisfies the EOM′ (1.8) and the condition that S satisfies the EOM (1.2).

The most important part is to calculate the inverse of

$$1 - S'\mathcal{V}' = \begin{pmatrix} (b/\beta)J^{-1} & -(\sqrt{2b}/\beta)[\boldsymbol{v}]^{\mathrm{T}} \\ -(\sqrt{2b}/\beta)S[\boldsymbol{v}] & 1 - S\mathcal{V} + (2/\beta)S\mathcal{U} \end{pmatrix}, \tag{3.1}$$

where we have defined

$$J = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad [\boldsymbol{v}] = \begin{pmatrix} \boldsymbol{v}_0 & \boldsymbol{v}_+ \\ \boldsymbol{v}_- & \boldsymbol{v}_0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} U_0 & U_+ \\ U_- & U_0 \end{pmatrix}. \tag{3.2}$$

Though in the original expression (1.8) $\mathcal{V}'$ is given as four blocks of "$\infty + 1$"-dimensional matrices with zero modes, here we have rearranged the rows and columns of the matrices so that the first row and the first column are associated with the zero modes, while the second row and the second column are with the non-zero modes.

The inverse of (3.1) can be evaluated by using the following formula which is similar to (B.5) in [15]:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}. \tag{3.3}$$

Though in the case of (B.5) in [15], all of A, B, C and D have to be square matrices, here A and D are square matrices but B and C can be rectangular ones. In the present case A is a 2×2 matrix and D is a $2\infty\times 2\infty$ one with

$$A^{-1}=\frac{\beta}{b}J,\quad B=-\frac{\sqrt{2b}}{\beta}[\boldsymbol{v}]^{\mathrm{T}},\quad C=-\frac{\sqrt{2b}}{\beta}S[\boldsymbol{v}],\quad D=1-S\mathcal{V}+\frac{2}{\beta}S\mathcal{U}. \tag{3.4}$$

With the use of the formula $[\boldsymbol{v}]J[\boldsymbol{v}]^{\mathrm{T}}=\mathcal{U}$, which can be derived using the relation

$$\boldsymbol{v}_0+\boldsymbol{v}_++\boldsymbol{v}_-=0, \tag{3.5}$$

we can calculate $CA^{-1}B$ easily: $CA^{-1}B=(2/\beta)S\mathcal{U}$. This implies that $D-CA^{-1}B$ is exactly equal to $1-S\mathcal{V}$. This observation will simplify our calculation tremendously.

Using this result we can write down $(1-S'\mathcal{V}')^{-1}$ without difficulty:

$$(1-S'\mathcal{V}')^{-1}=\begin{pmatrix}(1/b)\Big(\beta J+2J[\boldsymbol{v}]^{\mathrm{T}}(1-S\mathcal{V})^{-1}S[\boldsymbol{v}]J\Big) & \sqrt{2/b}J[\boldsymbol{v}]^{\mathrm{T}}(1-S\mathcal{V})^{-1}\\ \sqrt{2/b}(1-S\mathcal{V})^{-1}S[\boldsymbol{v}]J & (1-S\mathcal{V})^{-1}\end{pmatrix}. \tag{3.6}$$

Multiplying (3.6) by

$$(V'_+,V'_-)=\begin{pmatrix}b/(3\beta)(1,1) & \sqrt{2b}/\beta(\boldsymbol{v}_-^{\mathrm{T}},\boldsymbol{v}_+^{\mathrm{T}})\\ \sqrt{2b}/\beta(\boldsymbol{v}_+,\boldsymbol{v}_-) & (V_+-2U_+/\beta,V_--2U_-/\beta)\end{pmatrix}, \tag{3.7}$$

from the left, we find that

$$(V'_+,V'_-)(1-S'\mathcal{V}')^{-1}=\begin{pmatrix}(1,1) & (0,0)\\ \sqrt{2/b}X & (V_+,V_-)(1-S\mathcal{V})^{-1}\end{pmatrix}, \tag{3.8}$$

with X defined by $X=(\boldsymbol{v}_+,\boldsymbol{v}_-)J+(V_+,V_-)(1-S\mathcal{V})^{-1}S[\boldsymbol{v}]J$. Here we have used $(1,1)J[\boldsymbol{v}]^{\mathrm{T}}=-3(\boldsymbol{v}_-^{\mathrm{T}},\boldsymbol{v}_+^{\mathrm{T}})$ and $(\boldsymbol{v}_+,\boldsymbol{v}_-)J[\boldsymbol{v}]^{\mathrm{T}}=(U_+,U_-)$, which also follow from (3.5). Similarly, multiplication of (3.8) by $S'(V'_+,V'_-)^{\mathrm{T}}$ from the right can be easily performed if we note that

$$(\boldsymbol{v}_+,\boldsymbol{v}_-)J\begin{pmatrix}1\\1\end{pmatrix}=-3\boldsymbol{v}_0,\quad [\boldsymbol{v}]J\begin{pmatrix}\boldsymbol{v}_+^{\mathrm{T}}\\ \boldsymbol{v}_-^{\mathrm{T}}\end{pmatrix}=\begin{pmatrix}U_-\\U_+\end{pmatrix},\quad (\boldsymbol{v}_+,\boldsymbol{v}_-)J\begin{pmatrix}\boldsymbol{v}_+^{\mathrm{T}}\\ \boldsymbol{v}_-^{\mathrm{T}}\end{pmatrix}=U_0. \tag{3.9}$$

The result is

$$(V'_+,V'_-)(1-S'\mathcal{V}')^{-1}S'\begin{pmatrix}V'_-\\V'_+\end{pmatrix}=\begin{pmatrix}2b/(3\beta) & -\sqrt{2b}/\beta\boldsymbol{v}_0^{\mathrm{T}}\\ -\sqrt{2b}/\beta\boldsymbol{v}_0 & Y\end{pmatrix}, \tag{3.10}$$

with Y defined by

$$Y=(V_+,V_-)(1-S\mathcal{V})^{-1}S\begin{pmatrix}V_-\\V_+\end{pmatrix}+2U_0/\beta. \tag{3.11}$$

Adding V'_0 (1.9) to the RHS of (3.10) and equating it to S' (1.16), we find that only the non-zero-mode components give a non-trivial requirement of (1.2). Thus, our claim is proved.

4 EOM$'$ for the BCFT lump solutions

In this section, we would like to derive the b-independent form of the EOM$'$ (1.8) satisfied by the BCFT lump solution with S' given by (1.18). The most important part is again the calculation of the inverse of

$$1 - S'\mathcal{V}' = 1 - Q'\mathcal{V}' + \frac{1}{4\alpha}\boldsymbol{l}'_+\boldsymbol{l}'^{\mathrm{T}}_+\mathcal{V}' + \frac{1}{4\alpha}\boldsymbol{l}'_-\boldsymbol{l}'^{\mathrm{T}}_-\mathcal{V}', \tag{4.1}$$

where $\boldsymbol{l}'_+$ and $\boldsymbol{l}'_-$ are defined by

$$\boldsymbol{l}'_+ = \begin{pmatrix} \boldsymbol{l}' \\ \boldsymbol{l}' \end{pmatrix}, \quad \boldsymbol{l}'_- = \begin{pmatrix} \boldsymbol{l}' \\ -\boldsymbol{l}' \end{pmatrix}. \tag{4.2}$$

Using the formula

$$(M + \boldsymbol{v}_1\boldsymbol{w}_1^{\mathrm{T}} + \boldsymbol{v}_2\boldsymbol{w}_2^{\mathrm{T}})^{-1} = M^{-1} - \frac{M^{-1}\boldsymbol{v}_1\boldsymbol{w}_1^{\mathrm{T}}M^{-1}}{1 + \boldsymbol{w}_1^{\mathrm{T}}M^{-1}\boldsymbol{v}_1} - \frac{M^{-1}\boldsymbol{v}_2\boldsymbol{w}_2^{\mathrm{T}}M^{-1}}{1 + \boldsymbol{w}_2^{\mathrm{T}}M^{-1}\boldsymbol{v}_2}, \tag{4.3}$$

which is valid for $\boldsymbol{w}_1^{\mathrm{T}}M^{-1}\boldsymbol{v}_2 = \boldsymbol{w}_2^{\mathrm{T}}M^{-1}\boldsymbol{v}_1 = 0$, we obtain

$$\begin{aligned}(1 - S'\mathcal{V}')^{-1} &= (1 - Q'\mathcal{V}')^{-1} \\ &- \frac{1}{4\alpha}\frac{(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_+\boldsymbol{l}'^{\mathrm{T}}_+\mathcal{V}'(1 - Q'\mathcal{V}')^{-1}}{1 + \boldsymbol{l}'^{\mathrm{T}}_+\mathcal{V}'(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_+/(4\alpha)} - \frac{1}{4\alpha}\frac{(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_-\boldsymbol{l}'^{\mathrm{T}}_-\mathcal{V}'(1 - Q'\mathcal{V}')^{-1}}{1 + \boldsymbol{l}'^{\mathrm{T}}_-\mathcal{V}'(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_-/(4\alpha)},\end{aligned} \tag{4.4}$$

because of $\boldsymbol{l}'^{\mathrm{T}}_+\mathcal{V}'(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_- = 0$ which is due to the twist-even property of Q' and $\boldsymbol{l}'$. Hence, the EOM$'$ (1.8) is now put into the form

$$\begin{aligned}Q' - \frac{1}{2\alpha}\boldsymbol{l}'\boldsymbol{l}'^{\mathrm{T}} &= V'_0 + (V'_+, V'_-)(1 - Q'\mathcal{V}')^{-1}Q'\begin{pmatrix} V'_- \\ V'_+ \end{pmatrix} \\ &- \frac{1}{4\alpha}\frac{\boldsymbol{u}'_+\boldsymbol{u}'^{\mathrm{T}}_+}{1 + \boldsymbol{l}'^{\mathrm{T}}_+\mathcal{V}'(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_+/(4\alpha)} - \frac{1}{4\alpha}\frac{\boldsymbol{u}'_-\boldsymbol{u}'^{\mathrm{T}}_-}{1 + \boldsymbol{l}'^{\mathrm{T}}_-\mathcal{V}'(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_-/(4\alpha)},\end{aligned} \tag{4.5}$$

where $\boldsymbol{u}'_+$ and $\boldsymbol{u}'_-$ are defined by

$$\boldsymbol{u}'_+ = (V'_+, V'_-)(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_+, \quad \boldsymbol{u}'_- = (V'_+, V'_-)(1 - Q'\mathcal{V}')^{-1}\boldsymbol{l}'_-. \tag{4.6}$$

Since we have already calculated the first two terms on the RHS of (4.5) from the analysis in sec. 3 with S replaced by Q, let us consider the last two terms of (4.5). By using (3.8) with S replaced by Q, it is not difficult to see that $\boldsymbol{u}'_+$ and $\boldsymbol{u}'_-$ defined in (4.6) is reduced to

$$\boldsymbol{u}'_+ = \begin{pmatrix} 2\sqrt{b} \\ \boldsymbol{u}_+ \end{pmatrix}, \quad \boldsymbol{u}'_- = \begin{pmatrix} 0 \\ \boldsymbol{u}_- \end{pmatrix}, \tag{4.7}$$

with $\boldsymbol{u}_+$ and $\boldsymbol{u}_-$ being

$$\boldsymbol{u}_+ = 3\sqrt{2}(\boldsymbol{v}_+, \boldsymbol{v}_-)I_+ + 3\sqrt{2}(V_+, V_-)(1 - Q\mathcal{V})^{-1}Q[\boldsymbol{v}]I_+ + (V_+, V_-)(1 - Q\mathcal{V})^{-1}\boldsymbol{l}_+, \tag{4.8}$$

$$\boldsymbol{u}_- = \sqrt{2}(\boldsymbol{v}_+, \boldsymbol{v}_-)I_- + \sqrt{2}(V_+, V_-)(1-Q\mathcal{V})^{-1}Q[\boldsymbol{v}]I_- + (V_+, V_-)(1-Q\mathcal{V})^{-1}\boldsymbol{l}_-. \tag{4.9}$$

Here we have defined I_+, I_-, $\boldsymbol{l}_+$ and $\boldsymbol{l}_-$ as

$$I_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad I_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{l}_+ = \begin{pmatrix} \boldsymbol{l} \\ \boldsymbol{l} \end{pmatrix}, \quad \boldsymbol{l}_- = \begin{pmatrix} \boldsymbol{l} \\ -\boldsymbol{l} \end{pmatrix}. \tag{4.10}$$

Similarly, by explicit calculation of $\mathcal{V}'(1-Q'\mathcal{V}')^{-1}$, the inner products in the denominators of the last two terms in (4.5) are given as follows:

$$\begin{aligned} \boldsymbol{l}'^{\mathrm{T}}_+\mathcal{V}'(1-Q'\mathcal{V}')^{-1}\boldsymbol{l}'_+ &= b + 12V_{00} + 18I^{\mathrm{T}}_+[\boldsymbol{v}]^{\mathrm{T}}(1-Q\mathcal{V})^{-1}Q[\boldsymbol{v}]I_+ \\ &\quad + 6\sqrt{2}I^{\mathrm{T}}_+[\boldsymbol{v}]^{\mathrm{T}}(1-Q\mathcal{V})^{-1}\boldsymbol{l}_+ + \boldsymbol{l}^{\mathrm{T}}_+\mathcal{V}(1-Q\mathcal{V})^{-1}\boldsymbol{l}_+, \qquad (4.11) \\ \boldsymbol{l}'^{\mathrm{T}}_-\mathcal{V}'(1-Q'\mathcal{V}')^{-1}\boldsymbol{l}'_- &= -b + 4V_{00} + 2I^{\mathrm{T}}_-[\boldsymbol{v}]^{\mathrm{T}}(1-Q\mathcal{V})^{-1}Q[\boldsymbol{v}]I_- \\ &\quad + 2\sqrt{2}I^{\mathrm{T}}_-[\boldsymbol{v}]^{\mathrm{T}}(1-Q\mathcal{V})^{-1}\boldsymbol{l}_- + \boldsymbol{l}^{\mathrm{T}}_-\mathcal{V}(1-Q\mathcal{V})^{-1}\boldsymbol{l}_-. \qquad (4.12) \end{aligned}$$

Having simplified all the terms, let us turn to each component of (4.5). For the $(0,0)$-component, (4.5) is reduced to $\boldsymbol{l}'^{\mathrm{T}}_+\mathcal{V}'(1-Q'\mathcal{V}')^{-1}\boldsymbol{l}'_+ = 4\alpha$. Using (4.11), we find this becomes

$$\begin{aligned} 4\log(2t) &= 12V_{00} + 18I^{\mathrm{T}}_+[\boldsymbol{v}]^{\mathrm{T}}(1-Q\mathcal{V})^{-1}Q[\boldsymbol{v}]I_+ \\ &\quad + 6\sqrt{2}I^{\mathrm{T}}_+[\boldsymbol{v}]^{\mathrm{T}}(1-Q\mathcal{V})^{-1}\boldsymbol{l}_+ + \boldsymbol{l}^{\mathrm{T}}_+\mathcal{V}(1-Q\mathcal{V})^{-1}\boldsymbol{l}_+, \end{aligned} \tag{4.13}$$

which is explicitly b-independent. The $(n \geq 1, 0)$-component of (4.5) is

$$\boldsymbol{l} = \frac{1}{2}\boldsymbol{u}_+, \tag{4.14}$$

with $\boldsymbol{u}_+$ defined by (4.8) and is manifestly b-independent. Finally, the $(m \geq 1, n \geq 1)$-component of (4.5) reads

$$Q = V_0 + \begin{pmatrix} V_+ & V_- \end{pmatrix}(1-Q\mathcal{V})^{-1}Q\begin{pmatrix} V_- \\ V_+ \end{pmatrix} - \frac{\boldsymbol{u}_-\boldsymbol{u}_-^{\mathrm{T}}}{4\alpha + \boldsymbol{l}'^{\mathrm{T}}_-\mathcal{V}'(1-Q'\mathcal{V}')^{-1}\boldsymbol{l}'_-}. \tag{4.15}$$

The potential b-dependence is only in the denominator of the last term. However, from the explicit form of (4.12), we know the denominator is actually b-independent. To summarize, the b-independent form of the equation of motion for the BCFT lump solution is given by (4.13), (4.14) and (4.15).

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